Statistical properties of non-linear Froude-Krylov forces on cylinders

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ABSTRACT

The statistical properties of the second-order Froude-Krylov force on a cylinder (whether a vertical cylinder or a horizontal submerged cylinder), for narrow-band spectra, are investigated. For this purpose two families of stochastic processes are defined and for each family the probability density function and the probabilities of exceedance of the absolute maximum and of the absolute minimum are obtained. It is then proven that the above-mentioned Froude-Krylov force processes belong to these stochastic families.
The predictions for the Froude-Krylov force on a horizontal submerged cylinder agree with the results of a small-scale field experiment.

KEY WORDS: Wind-generated waves. Froude-Krylov force. Non-linearity effects. Probability of exceedance of the absolute maximum. Probability of exceedance of the absolute minimum. Vertical cylinder. Horizontal submerged cylinder.

INTRODUCTION

The amplitude of the wave force on a large structure may be obtained as the product of the Froude-Krylov force (which is defined as the force on the equivalent water volume) and the diffraction coefficient of the wave force (Sarpkaya and Isaacson, 1981). Therefore it is helpful for the design of large offshore structures to investigate the properties of the Froude-Krylov force.

According to the linear theory of wind-generated waves (Longuet-Higgins, 1963; Phillips, 1967) the linear Froude-Krylov force, whether on a vertical cylinder or on a horizontal submerged cylinder, represents a random Gaussian process of time. Therefore both the absolute maximum and the absolute minimum of the linear Froude-Krylov force have the same Rayleigh distribution, if the spectrum is very narrow (Longuet-Higgins, 1952).

Boccotti (2000) has shown that, for large horizontal cylinders, the two random processes *wave force on the solid cylinder*, and *Froude-Krylov wave force* have nearly the same very narrow spectrum, the same non-linearity effects, and the same statistical properties: equal distribution of the normalized crest-to-trough heights, distribution of the normalized absolute maximum and distribution of the normalized absolute minimum. This conclusion is based on the evidence of a small-scale field experiment which consisted in the real time comparison of the wave forces on a horizontal submerged cylinder and on an ideal equivalent water cylinder (see also Boccotti, 1996 and

Arena, 2002).

The statistics of non-linear wave forces was studied by Naess and Johnsen (1992), which proposed a numerical approach for calculating the probability density function of the second order hydrodynamic loads and response of compliant offshore structures.

In this paper an analytical formulation is proposed for narrow-band wind-generated wave processes (Tayfun, 1980), including the non-linear Froude-Krylov forces. In detail we define two families of non-linear stochastic processes ψ_1 and ψ_2: the first consisting of statistically symmetric processes, the second consisting of statistically non-symmetric processes. For each family of random processes we obtain the probability density function, the probability of exceedance of the absolute maximum and the probability of exceedance of the absolute minimum. For the family ψ_1 these properties depend upon one parameter δ, for the family ψ_2 they depend upon two parameters α_1 and α_2.

We prove that the horizontal component of the narrow-band second-order Froude-Krylov force (whether on a vertical cylinder or on a horizontal submerged cylinder) represents a random process of time which belongs to the stochastic family ψ_1, and the expression of parameter δ is derived for this process. We prove also that the vertical component of the narrow-band second-order Froude-Krylov force (on a horizontal submerged cylinder) represents a random process of time, which belongs to the stochastic family ψ_2 and the expressions of parameters α_1 and α_2 are obtained for this process.

Finally, we show that the analytical predictions for the Froude-Krylov force on a horizontal submerged cylinder agree with the conclusions of Boccotti (2000) based on experimental evidence.

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STATISTICAL PROPERTIES OF TWO STOCHASTIC FAMILIES WITH NARROW-BAND SPECTRUM

Let us define the two families of stochastic processes of time:

$$\psi_1(t) = f_1 a \sin[\chi(t)] + g_1 a^2 \sin[2\chi(t)] \,, \tag{1}$$

$$\psi_2(t) = f_2 a \cos[\chi(t)] + g_2 a^2 \cos^2[\chi(t)] + h_2 a^2 \sin^2[\chi(t)] \,, \tag{2}$$

where a is a Rayleigh distributed random variable, f_1, g_1, f_2, g_2, h_2 are parameters with some fixed values and where $\chi(t) = wt + \varphi$, with w the angular frequency and φ a random phase uniformly distributed in $(0, 2\pi)$.

The Probability Density Functions of the Stochastic Family ψ_1

Let us consider the normalized random process

$$\zeta_1 = (\psi_1 - \overline{\psi}_1)/\sigma_{\psi_1} \tag{3}$$

where $\overline{\psi}_1$ and σ_{ψ_1} are, respectively, the mean value and the standard deviation of random process ψ_1. Defining the two Gaussian random processes of time

$$Z_c(t) = a\cos[\chi(t)]/\sigma \,, \qquad Z_s(t) = a\sin[\chi(t)]/\sigma \tag{4}$$

where σ is the standard deviation of the linear process $a\sin[\chi(t)]$, the normalized process ζ_1 may be rewritten as

$$\zeta_1 = v Z_s + (\delta/v) Z_c Z_s \tag{5}$$

where

$$v = 1 \Big/ \sqrt{1 + \delta^2/4} \qquad \delta = 2 g_1 \sigma /|f_1| \,, \tag{6}$$

given that

$$\overline{\psi}_1 = 0 \,, \qquad \sigma_{\psi_1}^2 = \sigma^2\left(f_1^2 + \sigma^2 g_1^2\right). \tag{7}$$

The third and fourth moments of the family ζ_1, are given respectively by:

$$\overline{\zeta_1^3} = 0 \,, \qquad \overline{\zeta_1^4} = 3v^4 + 18\delta^2 + 9(\delta/v)^2 \,. \tag{8}$$

The fact the third moment is zero, suggests that ζ_1 is a symmetric process. To show this symmetry the probability density function of ζ_1 is derived. Firstly, by using Laplace transform technique, we evaluate the characteristic function of the process ζ_1, which is defined as the mean value of $e^{i\omega\zeta_1}$:

$$\overline{e^{i\omega\zeta_1}} = \exp\left[-\frac{1}{2}\omega^2\left(1 - \frac{\delta^2\omega^2}{v^2 + \omega^2\delta^2/v^2}\right)\right]\Bigg/\sqrt{v^2 + \omega^2\delta^2/v^2} \,. \tag{9}$$

The probability density function of ζ_1, which is obtained by inverse Fourier transform of $\overline{e^{i\omega\zeta}}$ (Eq. 9), is given by:

$$f_{\zeta_1}(\zeta) = \frac{1}{\pi}\int_0^{+\infty}\cos(\omega\zeta)\frac{\exp\left[-\frac{1}{2}\omega^2\left(1 - \frac{\delta^2\omega^2}{v^2 + \omega^2\delta^2/v^2}\right)\right]}{\sqrt{v^2 + \omega^2\delta^2/v^2}}\,d\omega \,. \tag{10}$$

Note that this probability density function is symmetric with respect to $\zeta = 0$, which means that the random process ζ_1 is statistically symmetric. Note also that the probability density function f_{ζ_1} (Eq. 10) of the normalized random variable ζ_1 approaches the probability density function of the normalized Gaussian variable, as δ

approaches zero. Fig. 1 compares the f_{ζ_1} for two values of δ and the probability density function of the normalized Gaussian random variable.

The Probability Density Functions of the Stochastic Family ψ_2

Let us consider the normalized random process

$$\zeta_2 = (\psi_2 - \overline{\psi}_2)/\sigma_{\psi_2} \tag{11}$$

where $\overline{\psi}_2$ and σ_{ψ_2} are, respectively, the mean value and the standard deviation of random process ψ_2. As function of the processes Z_c and Z_s (see Eq. 4), the process ζ_2 may be written as

$$\zeta_2 = \beta\left(Z_c + \alpha_1 Z_c^2 + \alpha_2 Z_s^2\right) - \beta(\alpha_1 + \alpha_2) \,, \tag{12}$$

where

$$\alpha_1 = \sigma\frac{g_2}{|f_2|} \,, \qquad \alpha_2 = \sigma\frac{h_2}{|f_2|} \,, \qquad \beta = \frac{1}{\sqrt{1 + 2(\alpha_1^2 + \alpha_2^2)}} \,. \tag{13}$$

The family ζ_2 is generally non-symmetric and its statistical properties have been deeply investigated by Arena & Fedele (2002), that derived the analytical expression of the probability density function f_{ζ_2} (obtained by applying the inverse Fourier transform to the characteristic function $\overline{e^{i\omega\zeta_2}}$). It is given by:

$$f_{\zeta_2}(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega\zeta}\exp\left[-\frac{1}{2}\frac{(\omega\beta)^2}{1 + 4(\omega\beta\alpha_1)^2}\right] \cdot$$

$$\cdot\frac{\exp\left\{-i\omega\beta\left[(\alpha_1 + \alpha_2) + \frac{(\omega\beta)^2\alpha_1}{1 + 4(\omega\beta\alpha_1)^2}\right]\right\}}{\sqrt{1 - 4(\omega\beta)^2\alpha_1\alpha_2 - 2i\,\omega\beta(\alpha_1 + \alpha_2)}}\,d\omega. \tag{14}$$

Let us observe that if the parameters α_1, α_2 approach zero, the non-linearity vanishes and the probability density function of ζ_2 (Eq. 14) tends to the Gaussian distribution. Fig. 2 shows the probability density function f_{ζ_2} (Eq. 14), for fixed values of (α_1, α_2). If the parameters α_1, α_2 approach zero, the non-linear effects vanish, and each process ζ belonging to the stochastic family (2), has to converge in probability to a Gaussian process.
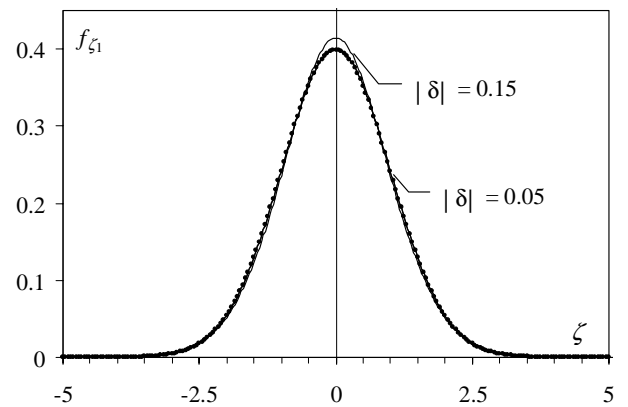

Fig. 1 Comparison between the probability density functions f_{ζ_1} (Eq. 10), for fixed values of δ, and the normalized Gaussian (dotted line).

The Probabilities of Exceedance of the Absolute Maximum and of the Absolute Minimum of the Stochastic Family ψ_1

Being the family ψ_1 (Eq. 1) symmetric, the distribution of the absolute maximum is equal to the distribution of the absolute minimum. Therefore we derive only the probability of exceedance of the absolute maximum.

If we rewrite the normalized process ζ_1 (Eq. 5) as

$$\zeta_1 = \frac{\nu a}{\sigma}\sin(\chi) + \frac{1}{2}\frac{\delta}{\nu}\left(\frac{a}{\sigma}\right)^2 \sin(2\chi) \qquad (15)$$

the first derivative $d\zeta_1/d\chi$ vanishes if

$$2\mu\cos^2\chi + \cos\chi - \mu = 0 \qquad (16)$$

where

$$\mu \equiv \delta a/(\nu^2\sigma). \qquad (17)$$

In the following we analyze the roots of Eq. 16 in the domain of the parameter μ. If $\mu \to 0$ the effects of non-linearity are negligible and Eq. 16 reduces itself to $\cos\chi = 0$: the abscissa $\chi = \pi/2$ of the maximum value is equal to the maximum abscissa for the linear process $\zeta_{1L} = (\nu a/\sigma)\sin(\chi)$. If $\mu \to \infty$ the non-linearity is predominant and Eq. 16 reduces itself to $\cos^2\chi = 1/2$: the abscissa of the maximum value is $\chi = \pi/4$. For finite μ, the abscissa of the maximum is within $\pi/4$ and $\pi/2$ and Eq. 16 is satisfied if

$$\cos\chi_{max} = \frac{4\mu}{1+\sqrt{1+32\mu^2}}, \qquad \sin\chi_{max} > 0. \qquad (18)$$

If we assume weak non-linear effects (that is $\mu << 1$), for a fixed value of a/σ, Eq. 18 may be expanded in Taylor series as

$$\cos\chi_{max} \cong 2\mu - 16\mu^2 \qquad \sin\chi_{max} \cong 1 - 2\mu^2. \qquad (19)$$

Substitution of expressions 19 in Eq. 15, after some algebra, by retaining only the lower order terms, gives us the approximate expression for the maximum

$$\zeta_{1max} = \zeta_1(\chi_{max}) \cong \nu u + \frac{\delta^2}{2\nu^3}u^3. \qquad (20)$$

Successive approximations procedure yields the following expressions for u_0 such that Eq. 20 is satisfied

$$u_0 \cong \frac{\zeta_{1max}}{\nu} - \frac{\delta^2}{2\nu^7}\zeta_{1max}^3. \qquad (21)$$

Finally, having the variable u the Rayleigh distribution [that is $P(u \geq z) = \exp(-z^2/2)$], the probability of exceedance for the absolute maximum has expression

$$P(\zeta_{1max} > \zeta) = \exp\left[-\frac{1}{2}\zeta^2\left(\frac{1}{\nu} - \frac{\delta^2}{2\nu^7}\zeta^2\right)^2\right]. \qquad (22)$$

This probability of exceedance, for fixed values of δ, is shown in Fig. 3. Let us note that the deviation from the Rayleigh distribution is weak for $|\delta| < 0.05$.

The Probabilities of Exceedance of the Absolute Maximum and of the Absolute Minimum of the Stochastic Family ψ_2

Supposing that $f_2 > 0$, from Eq. 2 we obtain the amplitudes of the absolute maximum and of the absolute minimum (in absolute value), which are given respectively by:

$$\Psi_{high} = f_2(x,y)a + g_2(x,y)a^2, \quad \Psi_{low} = f_2(x,y)a - g_2(x,y)a^2. \qquad (23)$$

Arena and Fedele (2002) showed that if the condition

$$\frac{|f_2(x,y)|}{|g_2(x,y) - h_2(x,y)|a/2} > 4 \qquad (24)$$

is satisfied, the probabilities of exceedance $P(\zeta_{2max} > \zeta)$ of the absolute maximum and $P(\zeta_{2min} > \zeta)$ of the absolute minimum of the dimensionless variables

$$\zeta_{2max} = \frac{\Psi_{2max}}{\sigma_{\psi 2}} = u\beta + \alpha_1\beta u^2, \quad \zeta_{2min} = \frac{\Psi_{2min}}{\sigma_{\psi 2}} = u\beta - \alpha_1\beta u^2 \qquad (25)$$

where the random variable u has Rayleigh distribution, are given by:

$$\text{if } \alpha_1 > 0 \quad \begin{cases} P(\zeta_{2max} > \zeta) = f_a(\zeta), \\ P(\zeta_{2min} > \zeta) = \begin{cases} f_b(\zeta) & \text{if } \zeta \leq \beta/(4|\alpha_1|), \\ 0 & \text{if } \zeta > \beta/(4|\alpha_1|), \end{cases} \end{cases} \qquad (26)$$

$$\text{if } \alpha_1 < 0 \quad \begin{cases} P(\zeta_{2max} > \zeta) = \begin{cases} f_b(\zeta) & \text{if } \zeta \leq \beta/(4|\alpha_1|), \\ 0 & \text{if } \zeta > \beta/(4|\alpha_1|), \end{cases} \\ P(\zeta_{2min} > \zeta) = f_a(\zeta), \end{cases} \qquad (27)$$

where the functions f_a and f_b are respectively:

$$f_a(\zeta) = \exp\left[-\left(1 - \sqrt{1+4|\alpha_1|\zeta/\beta}\right)^2 \Big/ \left(8\alpha_1^2\right)\right], \qquad (28)$$

$$f_b(\zeta) = \exp\left[-\left(1 - \sqrt{1-4|\alpha_1|\zeta/\beta}\right)^2 \Big/ \left(8\alpha_1^2\right)\right] +$$

$$-\exp\left[-\left(1 + \sqrt{1-4|\alpha_1|\zeta/\beta}\right)^2 \Big/ \left(8\alpha_1^2\right)\right] \qquad (29)$$

(the parameters α_1, α_2 and β are defined by Eq. 13).

The minimum probability that the condition (24) is satisfied has expression as the following

$$P_l = \exp\left[-\frac{1}{8(\alpha_1 - \alpha_2)^2}\right]. \qquad (30)$$

The probability P_l may be interpreted as the fraction of the realizations of the non-linear process ψ in which condition (24) is not verified. Let observe that for $|\alpha_1 - \alpha_2| \leq .135$ the probability P_l is close to 1/1000. Fig. 4 shows the probabilities of exceedance $P(\zeta_{2max} > \zeta)$ of the absolute maximum and $P(\zeta_{2min} > \zeta)$ of the absolute minimum for fixed values of α_1 (and for $|\alpha_2| = \alpha_1$). Observe that for α_1 approaching zero both the probabilities of exceedance reduce themselves to the Rayleigh distribution. For $\alpha_1 \neq 0$ the two distributions are different: in particular for a fixed threshold of the probability of exceedance, if $\alpha_1 > 0$ the absolute
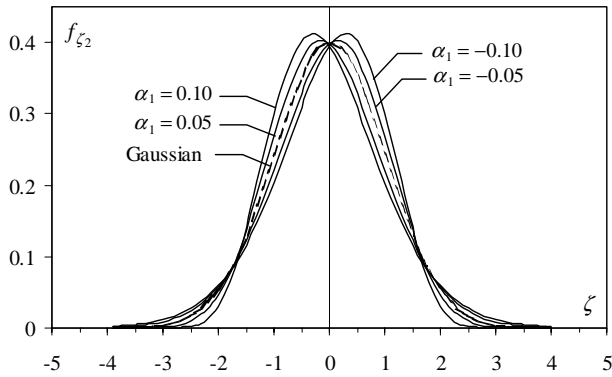
Fig. 2 The probability density functions f_{ζ_2} (Eq. 14), for fixed values of α_1 (it has been assumed $\alpha_2 = -\alpha_1$). The dashed line is the Gaussian distribution, obtained for $\alpha_1 = 0$.
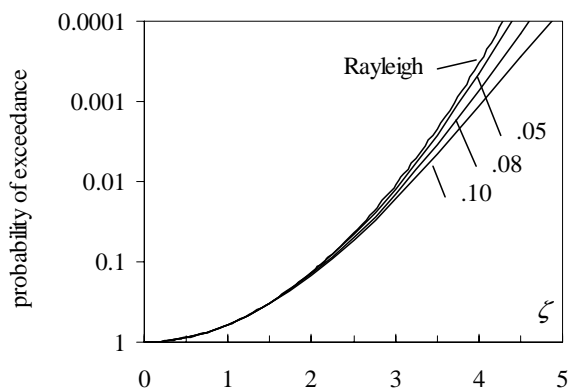

Fig. 3 The probability of exceedance $P(\zeta_{1\max} > \zeta)$ of the absolute maximum (Eq. 22), for fixed $|\delta|$. The probability $P(\zeta_{1\max} > \zeta)$ is equal to the Rayleigh distribution for $\delta = 0$.

maximum is greater than the absolute minimum [and therefore each realization of the process is a sequence of waves which have crest amplitude (absolute maximum) greater than the trough amplitude (absolute minimum)]; if $\alpha_1 < 0$ the absolute minimum is greater than the absolute maximum (and therefore each realization of the process is a sequence of waves which have the trough amplitude greater than the crest amplitude). It is also easy to verify that the distributions of Fig. 4 are not modified if α_2 ranges between $-|\alpha_1|$ and $|\alpha_1|$.

APPLICATIONS

Let us assume the reference frame (x, y, z) with the x-axis horizontal (direction along which the waves attack), the y-axis horizontal and the z-axis vertical with origin at the mean water level, as well as d the bottom depth.

The narrow-band second-order Froude-Krylov force processes, both for a vertical cylinder and for a horizontal submerged cylinder, are derived by analytical integration of the narrow-band second-order wave pressure (see Boccotti 2000).

The steepness ε (being $\varepsilon = k\sigma$, k the wave number and σ the standard deviation of the linear surface displacement) ranges typically between 0.05 and 0.08.
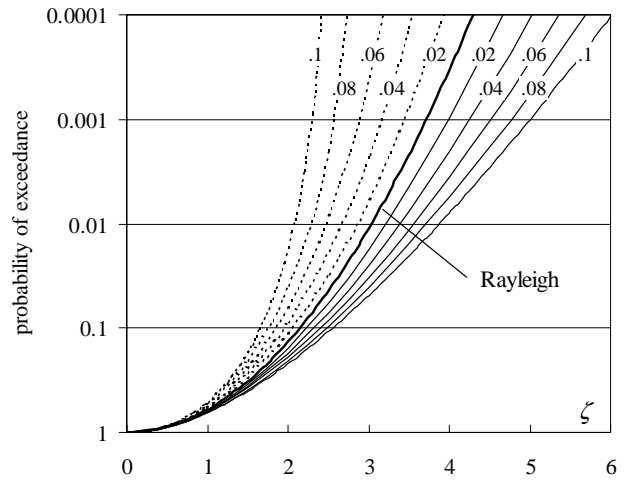

Fig. 4 Probabilities of exceedance for fixed values of $|\alpha_1|$ (assuming that $|\alpha_2| = |\alpha_1|$). (i) Positive α_1: continuous lines are the absolute maximum distribution $P(\zeta_{2\max} > \zeta)$, dashed lines are the absolute minimum distribution $P(\zeta_{2\min} > \zeta)$. (ii) Negative α_1: continuous lines are the $P(\zeta_{2\min} > \zeta)$, dashed lines are the $P(\zeta_{2\max} > \zeta)$.

The Narrow-Band Second-Order Froude-Krylov Force on a Vertical Cylinder

The sectional force

Let us consider an ideal water vertical cylinder with radius R (see Fig. 5). The Froude-Krylov force (force for unitary length), at a fixed level z, is given by

$$F_x(z) = -\int_0^{2\pi} R\, \Delta p(R, \theta, z) \cos(\theta)\, d\theta \tag{31}$$

(we consider the variable transformation $x = r\cos\theta$; $y = r\sin\theta$ - see Fig. 5). By integrating we obtain the narrow-band horizontal component of the Froude-Krylov force

$$F_x(z) = -\rho\, g\, 2\pi\, R\, a\, J_1(kR) \frac{\cosh[k(z+d)]}{\cosh(kd)} \sin(wt + \varphi) - 2\pi\rho\, g\ \cdot$$
$$\cdot\, kRa^2 J_1(2kR) \frac{3\cosh[2k(z+d)] - \sinh^2(kd)}{4\sinh^3(kd)\cosh(kd)} \sin[2(wt + \varphi)], \tag{32}$$

that belongs to the symmetric family ψ_1 with parameter

$$\delta = -\varepsilon \frac{J_1(2kR)}{|J_1(kR)|} \frac{3\cosh[2k(z+d)] - \sinh^2(kd)}{2\sinh^3(kd)\cosh[k(z+d)]} \tag{33}$$

where $J_1(x)$ is the Bessel function of the first kind.

The total force

The total force, which is defined as

$$F_x = \int_{-h}^{0} F_x(z)\, dz, \tag{34}$$

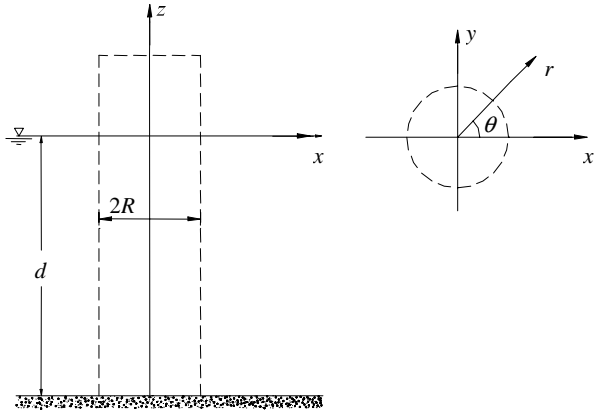belongs to the family ψ_1 (Eq. 1) too, with parameter

Fig. 5 Froude-Krylov force on a vertical cylinder: the reference frame.

$$\delta = -\varepsilon \frac{J_1(2kR)}{|J_1(kR)|} \frac{3\sinh(2kd) - 2kd\sinh^2(kd)}{4\sinh^4(kd)} . \quad (35)$$

In Fig. 6 the parameter δ/ε as function of kR is shown, for fixed values of kd. Let us note that the parameter δ decreases as the depth increases; it is also small: for $kd > 1.5$ we have $|\delta| < 0.5\varepsilon$ (that is $|\delta| < 0.025 \div 0.04$).

Therefore the second-order Froude-Krylov force on a vertical cylinder for a narrow-band spectrum, is a symmetric quasi-Gaussian process; as a consequence the probabilities of exceedance of the absolute maximum and of the absolute minimum are very close to the Rayleigh distribution (compare to Fig. 3).

The Narrow-Band Second-Order Froude-Krylov Force on a Horizontal Submerged Cylinder

Let us consider an ideal water-horizontal cylinder with radius R and centre at the level $z = -h$. The Froude-Krylov force components (forces for unitary length) are respectively

$$F_x = -\int_0^{2\pi} R\Delta p(R,\theta)\cos(\theta)\mathrm{d}\theta, \quad F_z = -\int_0^{2\pi} R\Delta p(R,\theta)\sin(\theta)\mathrm{d}\theta \quad (36)$$

where (r,θ) define polar co-ordinates ($x = r\cos\theta$, $z = -h + r\sin\theta$ - see Fig. 7).

The horizontal component F_x

From Eq. 36, after analytical integration, we obtain the narrow-band second-order horizontal component F_x

$$F_x = -\rho g \pi kR^2 a \frac{\cosh[k(d-h)]}{\cosh(kd)}\sin(wt+\varphi) - \rho g\, 2\pi kR a^2 \cdot$$

$$\left[kR\frac{3\cosh[2k(d-h)]}{4\sinh^3(kd)\cosh(kd)} - \frac{J_1(2kR)}{4\sinh(kd)\cosh(kd)}\right]\sin[2(wt+\varphi)]. \quad (37)$$

It is easy to verify that the process force F_x (Eq. 37) belongs to the stochastic family ψ_1 (Eq. 1), with parameter

$$\delta = -\varepsilon \frac{\dfrac{3\cosh[2k(d-h)]}{\sinh^2(kd)} - \dfrac{J_1(2kR)}{kR}}{\cosh[k(d-h)]\sinh(kd)} . \quad (38)$$
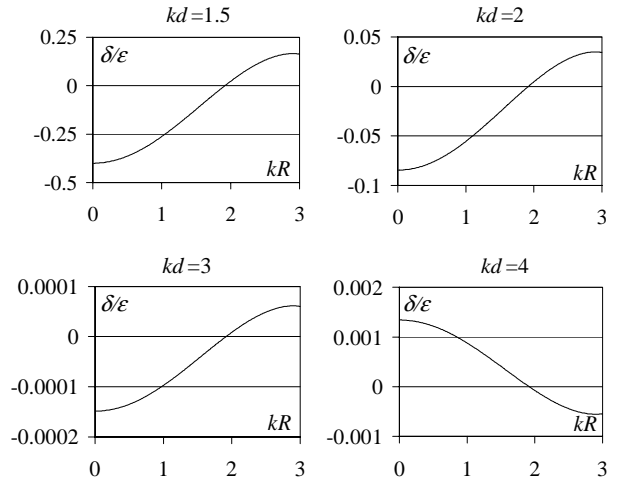

Fig. 6 Froude-Krylov force on a vertical cylinder: the parameter δ/ε as function of kR, for fixed values of kd.
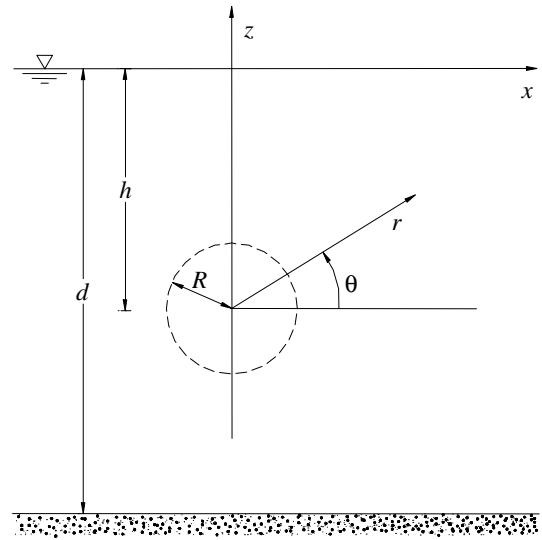

Fig. 7 Froude-Krylov force on a horizontal submerged cylinder: the reference frame.

In Fig. 8 the parameters δ/ε as function of kR are shown, for fixed values of h/d and kd. Let us note that for fixed radius R the parameter δ decreases as the depth d increases. The non-linear effects are weak (the parameter $|\delta|$ is smaller than 0.05), so that the process F_x (Eq. 37) may be considered symmetric quasi-Gaussian and the probabilities of exceedance of the absolute maximum and of the absolute minimum (Eq. 22) are very close to the Rayleigh distribution (Fig. 3).

The vertical component F_z

The narrow-band vertical component F_z has the following expression

$$F_z = -\rho g \pi kR^2 a \frac{\sinh[k(d-h)]}{\cosh(kd)}\cos(wt+\varphi) - \rho g\, 2\pi kR a^2 \cdot$$

$$\cdot\left\{\frac{3k\sinh[2k(d-h)]}{4\sinh^3(kd)\cosh(kd)}\cos[2(wt+\varphi)] - \frac{\sinh[2k(d-h)]}{2\sinh(2kd)}I_1(2kR)\right\} \quad (39)$$
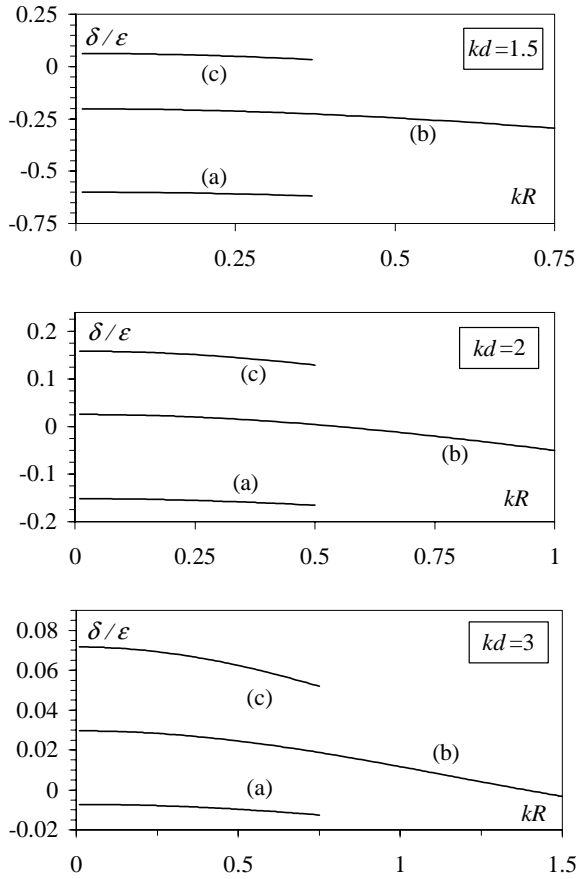
Fig. 8 Horizontal component of the Froude-Krylov force on a horizontal submerged cylinder. The parameter δ/ε as function of kR, for fixed values of kd: (a) $h/d = 0.25$; (b) $h/d = 0.50$; (c) $h/d = 0.75$.

in which $I_1(x)$ is the modified Bessel function of first kind. This process belongs to the stochastic family ψ_2 (Eq. 2) with parameters

$$\alpha_1 = \varepsilon \frac{\cosh[k(d-h)]}{\sinh(kd)} \left[-\frac{3}{\sinh^2(kd)} + \frac{I_1(2kR)}{kR} \right]$$

$$\alpha_2 = \varepsilon \frac{\cosh[k(d-h)]}{\sinh(kd)} \left[\frac{3}{\sinh^2(kd)} + \frac{I_1(2kR)}{kR} \right].$$

(40)

In this case the process is non-symmetric. In Fig. 9 the parameters α_1 and α_2 as function of kd are showed, for fixed values of h/R and kR. Let us note that for a fixed depth kd and a fixed kh, α_1 increases as the radius kR increases; for fixed kd and kR, α_1 decreases as kh increases (that is as the cylinder tends to approach the bottom).

Furthermore, from Fig. 9 we observe that maximum values of α_1 and α_2 are within 0.05 and 0.08 (being the maximum value of α_1/ε and α_2/ε very close to 1).

Finally, being $\alpha_1 > 0$ (in the ranges of kd, kR and h/R values considered in Fig. 9), the probabilities of exceedance of the positive peak (absolute maximum) of F_z (Eq. 39) and of the negative peak (absolute minimum) of F_z are different: in particular for a fixed
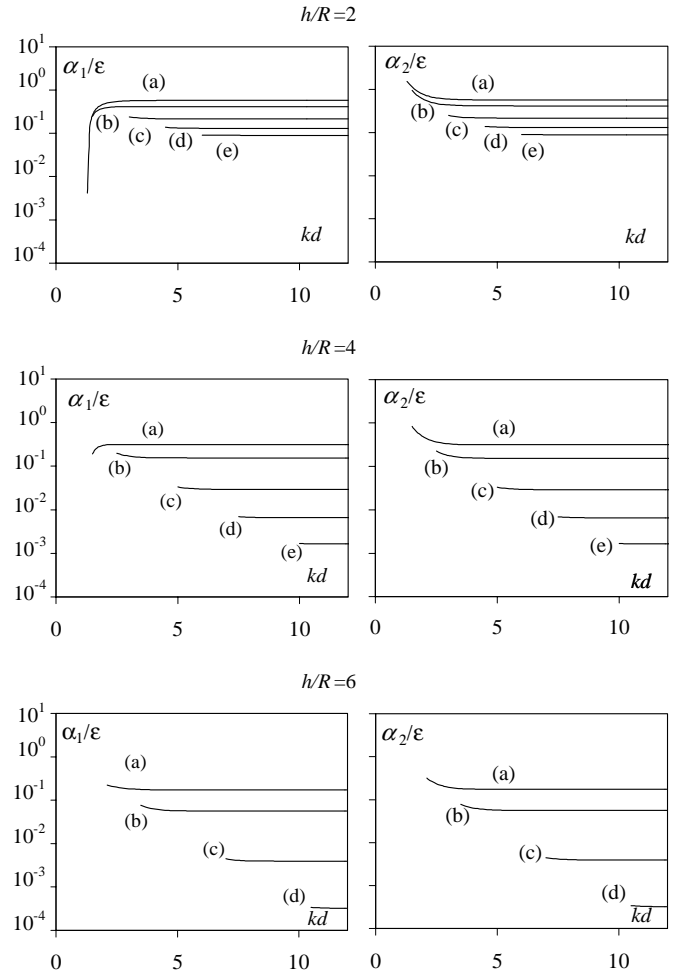

Fig. 9 Vertical component of the Froude-Krylov force on a horizontal submerged cylinder: the parameters α_1/ε and α_2/ε as function of kd, for fixed values of h/R. (a) $kR = .3$; (b) $kR = .5$; (c) $kR = 1$; (d) $kR = 1.5$; (e) $kR = 2$.

threshold of probability of exceedance, the crest (positive peak) of the wave force F_z is greater than the trough (negative peak). In words for $\alpha_1 > 0$ each realization of the process Froude-Krylov vertical force F_z is a sequence of waves, which have crest amplitude greater than trough amplitude (Fig. 4).

COMPARISON WITH EXPERIMENTAL DATA

To check our results we have resorted to the file data of a small-scale field experiment (Boccotti, 1996; Arena, 2002), which is relevant to the forces on a horizontal submerged cylinder.

Let us start with the horizontal component of the Froude-Krylov force. We have estimated parameter δ by means of Eq. 38 from the data set of this experiment: the peak period (being necessary to obtain wave number k), root mean square surface displacement (being necessary to obtain ε), water depth d, submergence h of the cylinder centre, radius R of the horizontal cylinder. We have evaluated the value of δ for each record of the experiment, and these values prove to range between –0.05 and 0.01. In our analytical approach we have shown that the probability of exceedance of the

absolute maximum and the probability of exceedance of the absolute minimum are equal to each other and are given by Eq. 22. With $|\delta|$ within 0.05 as in the experiment we are dealing with, Eq. 2 is very close to the Rayleigh form (see Fig. 3).

Hence we can expect that both the probability of exceedance of the absolute maximum and the probability of exceedance of the absolute minimum of the horizontal Froude-Krylov force are very close to each other and are very close to the Rayleigh form. This is what actually occurs, and can be appreciated from Fig. 10 (data by Boccotti, 2000).

Let us pass to the vertical component of the Froude-Krylov force. We have estimated the pair α_1, α_2 for each record of the experiment by means of Eq. 40. Parameter α_1 proves to range between 0.011 and 0.027, and parameter α_2 between 0.011 and 0.041. For each pair α_1, α_2 we have obtained the probability of exceedance of the absolute maximum and the probability of exceedance of the absolute minimum by means of Eq. 26. The two extreme probabilities of exceedance for the set of pairs α_1, α_2 are shown in Fig. 11 (lines a and b). [The upper panel is relevant to the absolute maximum and the lower panel to the absolute minimum.] We see that the probability of exceedance of the absolute maximum (positive peak of F_z) is greater than the probability of exceedance based on the Rayleigh form. On the contrary, the probability of exceedance of the absolute minimum (negative peak of F_z) is smaller than the probability of exceedance based on the Rayleigh form. The data points for the probability of exceedance are those of Boccotti (2000): they are relevant to the whole set of record during the experiment, and they clearly confirm the trend of our theoretical predictions.

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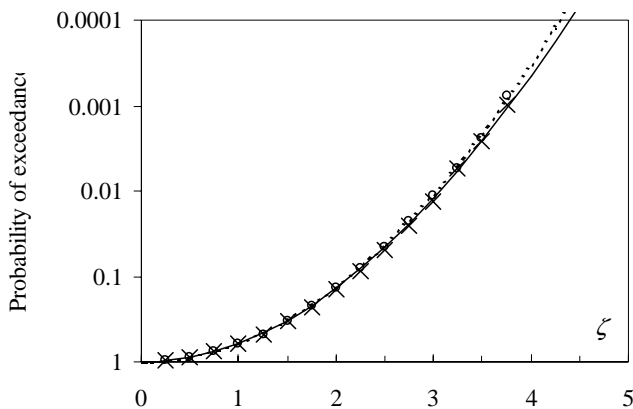Tayfun, MA (1980). "Narrow-band nonlinear sea waves". *J. Geophys. Res.*, Vol 85, pp 1548-1552.

Fig. 10 The distributions of the positive and negative peaks of the horizontal Froude-Krylov force on a horizontal submerged cylinder: (o) positive peaks data; (x) negative peaks data (from Boccotti, 2000). Theoretical predictions are obtained from Eq. 22 for |δ|=0.01 (dotted line) and 0.05 (continuous line) , which are the minimum and the maximum |δ| values in the experimental range of kR , kh and kd , assuming $\varepsilon = 0.06$.
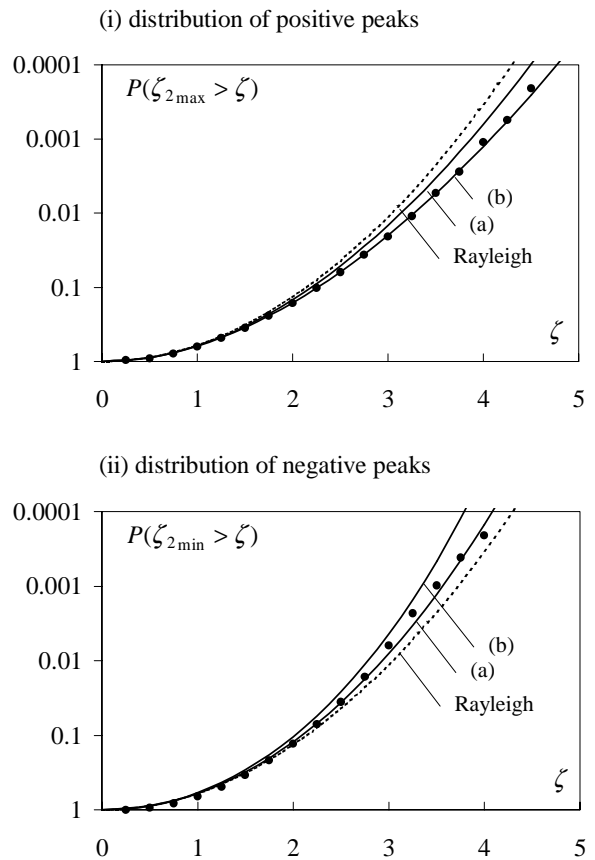

Fig. 11 The distributions of the peaks of the vertical Froude-Krylov force on a horizontal submerged cylinder: (i) positive (upward) peak; (ii) negative (downward) peak. Continuous lines are the predictions from Eq. 43: lines (a) are obtained for $\alpha_1 = 0.011$, lines (b) are obtained for $\alpha_1 = 0.041$, which are the minimum and the maximum α_1 in the experimental range of kR , kh and kd , assuming $\varepsilon = 0.06$ (the corresponding α_2 are equal to 0.011 and 0.041 respectively).

Points showed experimental data (from Boccotti, 2000).